Transmission problems with sign-changing coefficients

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Usual partial differential equations arising in physics involve coefficients describing the properties of the medium, like the density and the Lamé coefficients in elasticity, the dielectric permittivity and magnetic permeability in electromagnetism, or the thermal conductivity for the heat equation. Classically, these coefficients are supposed to be strictly positive. This hypothesis plays an essential role in the mathematical and numerical analysis. In particular, the positivity of the coefficients is required to establish the coercivity of the related bilinear form, needed to apply the celebrated Lax-Milgram and Céa theorems.

But this classical approach fails if the coefficients in the equation are sign-changing. What happens in that case? What can we say concerning well-posedness and convergence of Galerkin methods? This is the subject of this presentation.

First, I will present two examples of such sign-changing problems. The first one is related to the study of plasmonic waves that propagate at the interface between a dielectric and a metal. The second one arises in the analysis of the Linear Sampling Method for inverse scattering.

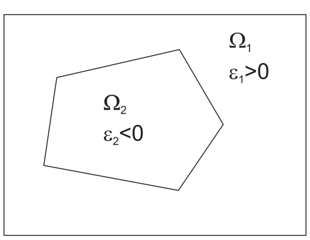
More precisely, let Ω be a bounded domain of $\mathbb{R}^N$ which is the union of two subdomains Ω_1 and Ω_2, and let ε in $L^\infty(\Omega)$ (with $1/\varepsilon \in L^\infty(\Omega)$) be a function which takes positive values in Ω_1 and negative values in Ω_2. The two problems that we will consider are of the classical form

$$\text{Find } u \in V \text{ such that } a(u,v) = \ell(v) \quad \forall v \in V. \tag{1}$$

The first problem denoted (P^∇) corresponds to $a(u,v) = \displaystyle\int_\Omega \varepsilon \nabla u \cdot \nabla v$ and $V = H^1_0(\Omega)$.

The second problem denoted (P^Δ) corresponds to $a(u,v) = \displaystyle\int_\Omega \varepsilon \Delta u \, \Delta v$ and $V = H^2_0(\Omega)$.

For the analysis, we will use the following simple idea: problem (1) is well-posed if and only if there exists an isomorphism T of V such that the bilinear form $a(u, Tv)$ is coercive on V. The question becomes the following: how can we build such operators T? Two different approaches will be used for problems (P^∇) and (P^Δ). This will allow us to establish conditions on ε ensuring that problems (P^∇) or (P^Δ) respectively are well-posed, or at least of Fredholm type. The results are very different between the two problems.



For instance, for the configuration of this figure (2D case with a piecewise constant ε), (P^Δ) is always of Fredholm type, while (P^∇) is of Fredholm type if and only if

$$\frac{\varepsilon_1}{\varepsilon_2} \notin \left[\frac{\Phi - 2\pi}{\Phi}, \frac{\Phi}{\Phi - 2\pi} \right]$$

where Φ is the smallest angle of the polygonal interface.

References

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